GALOIS COHOMOLOGY OF REAL SEMISIMPLE GROUPS

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Abstract. Let $\mathbf{G}$ be a connected, compact, semisimple algebraic group over the field of real numbers $\mathbb{R}$. Using Kac diagrams, we describe combinatorially the first Galois cohomology sets $H^1(\mathbb{R}, \mathbf{H})$ for all inner forms $\mathbf{H}$ of $\mathbf{G}$. As examples, we compute explicitly H^1 for all real forms of the simply connected simple group of type $\mathbf{E}_7$ (which has been known since 2013) and for all real forms of half-spin groups of type $\mathbf{D}_{2k}$ (which seems to be new).

0. Introduction

Let $\mathbf{H}$ be a linear algebraic group defined over the field of real numbers $\mathbb{R}$. For the definition of the first (nonabelian) Galois cohomology set $H^1(\mathbb{R}, \mathbf{H})$ see Section 4 below. Galois cohomology can be used to answer many natural questions (on classification of real forms, on the connected components of the set of $\mathbb{R}$-points of a homogeneous space etc.). The Galois cohomology sets $H^1(\mathbb{R}, \mathbf{H})$ of the classical groups are well known. Recently the sets $H^1(\mathbb{R}, \mathbf{H})$ were computed for "most" of the *simple* $\mathbb{R}$-groups by Adams [A], in particular, for all *simply connected* simple $\mathbb{R}$-groups by Adams [A] and by Borovoi and Evenor [BE].

Victor G. Kac [K] used what was later called Kac diagrams (see Onishchik and Vinberg [OV2, Sections 3.3.7 and 3.3.11]) to classify the conjugacy classes of automorphisms of finite order of a simple Lie algebra over the field of complex numbers $\mathbb{C}$. Let $\mathbf{G}$ be a *compact* (anisotropic), simply connected, simple algebraic group over $\mathbb{R}$. Write $\mathbf{G}_\mathbb{C} = \mathbf{G} \times_\mathbb{R} \mathbb{C}$, $\mathfrak{g}_\mathbb{C} = \mathrm{Lie}\,(\mathbf{G}_\mathbb{C})$. With this notation, Kac classified the conjugacy classes of elements of order n in $\mathrm{Aut}\,\mathfrak{g}_\mathbb{C} = \mathrm{Aut}\,\mathbf{G}_\mathbb{C}$. In particular, he classified the conjugacy classes of elements of order n in the group of inner automorphisms $\mathbf{G}^{\mathrm{ad}}(\mathbb{C}) \subset \mathrm{Aut}\,\mathbf{G}_\mathbb{C}$, where $\mathbf{G}^{\mathrm{ad}} := \mathbf{G}/\mathbf{Z}_\mathbf{G}$ is the corresponding adjoint group. Equivalently, he classified the conjugacy classes of elements of order n in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$.

Note that the set of conjugacy classes of elements of order $n = 2$ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ is in canonical bijection with the first Galois cohomology set $H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$, see Serre [S, Section III.4.5, Theorem 6]. Thus Kac computed $H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$, the Galois cohomology of the compact, simple, *adjoint* $\mathbb{R}$-group $\mathbf{G}^{\mathrm{ad}}$.

In the present paper we use the method of Kac diagrams in order to compute $H^1(\mathbb{R}, \mathbf{G})$, or more generally $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$, where $\mathbf{G}$ is a connected, compact, *semisimple* $\mathbb{R}$-group, not necessarily adjoint, and ${}_{\boldsymbol{q}}\mathbf{G}$ is the *inner* twisted form of $\mathbf{G}$ corresponding to a Kac diagram $\boldsymbol{q}$. This is reduced

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to classifying conjugacy classes of square roots of a given central element $z = z_{\boldsymbol{q}} \in \mathbf{G}(\mathbb{R})$.

The plan of the paper is as follows. In Section 1 we introduce the necessary notation. In Section 2 we describe, following Bourbaki [Bou], the action of $P^\vee/Q^\vee$ on the extended Dynkin diagram of a root system R, where $P^\vee$ is the coroot lattice and $Q^\vee$ is the coweight lattice. The heart of the paper is Section 3, where we prove Theorem 3.4 describing the conjugacy classes of n-th roots of a given central element z in a connected semisimple compact Lie group G in terms of certain combinatorial objects called *Kac n-labelings of the extended Dynkin diagram $\widetilde{D}$ of G.* Using this theorem (in the case $n = 2$) and a result of [B1], in Section 4 we prove Theorem 4.3, which is the main result of this paper. It describes the first Galois cohomology set $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ of an inner twisted form ${}_{\boldsymbol{q}}\mathbf{G}$ of a connected compact (anisotropic) semisimple $\mathbb{R}$-group $\mathbf{G}$ in terms of Kac 2-labelings. As an example, in Section 5 we compute, using Kac 2-labelings, the Galois cohomology sets $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ for all $\mathbb{R}$-forms ${}_{\boldsymbol{q}}\mathbf{G}$ of the compact simply connected group $\mathbf{G}$ of type $\mathbf{E}_7$; these results were obtained earlier by other methods in [A] and [BE], see also Conrad [C, Proof of Lemma 4.9]. As another example, in Section 6 we compute the Galois cohomology sets $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ for all $\mathbb{R}$-forms of a half-spin compact group of type $\mathbf{D}_\ell$ for even $\ell > 4$; these results seem to be new.

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1. Notation

In this paper $\mathbf{G}$ always is a connected, compact (anisotropic), semisimple algebraic group over the field of real numbers $\mathbb{R}$. We write $\mathbf{Z}_{\mathbf{G}}$ for the center of $\mathbf{G}$. Let $\mathbf{G}^{\mathrm{ad}} = \mathbf{G}/\mathbf{Z}_{\mathbf{G}}$ denote the corresponding adjoint group, and let $\mathbf{G}^{\mathrm{sc}}$ denote the universal covering of $\mathbf{G}$ (which is simply connected). Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. We denote by $\mathfrak{t}$ the Lie algebra of $\mathbf{T}$, which is a vector space over $\mathbb{R}$. Let $\mathbf{N} = \mathcal{N}_{\mathbf{G}}(\mathbf{T})$ denote the normalizer of $\mathbf{T}$ in $\mathbf{G}$. Let $\mathbf{W} = \mathbf{N}/\mathbf{T}$ be the Weyl group, which is a finite algebraic group.

Let $\mathbf{T}^{\mathrm{ad}} := \mathbf{T}/\mathbf{Z}_{\mathbf{G}}$ be the image of $\mathbf{T}$ in $\mathbf{G}^{\mathrm{ad}}$, and let $\mathbf{T}^{\mathrm{sc}}$ denote the preimage of $\mathbf{T}$ in $\mathbf{G}^{\mathrm{sc}}$. Then $\mathbf{T}^{\mathrm{ad}}$ is a maximal torus in $\mathbf{G}^{\mathrm{ad}}$, and $\mathbf{T}^{\mathrm{sc}}$ is a maximal torus in $\mathbf{G}^{\mathrm{sc}}$. Set

$$X = \mathsf{X}(\mathbf{T}_{\mathbb{C}}) := \mathrm{Hom}(\mathbf{T}_{\mathbb{C}}, \mathbb{G}_{m,\mathbb{C}}), \quad X^\vee = \mathsf{X}^\vee(\mathbf{T}_{\mathbb{C}}) := \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, \mathbf{T}_{\mathbb{C}}),$$

where $\mathbf{T}_{\mathbb{C}} = \mathbf{T} \times_{\mathbb{R}} \mathbb{C}$ and $\mathbb{G}_{m,\mathbb{C}}$ is the multiplicative group over $\mathbb{C}$; then X and $X^\vee$ are the character group and the cocharacter group of $\mathbf{T}_{\mathbb{C}}$, respectively.

We have a canonical isomorphism of abelian complex Lie groups

$$X^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \xrightarrow{\sim} \mathbf{T}(\mathbb{C}), \quad \chi \otimes u \mapsto \chi(u), \quad \chi \in X^\vee,\ u \in \mathbb{C}^\times = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}).$$

Thus we obtain an isomorphism of abelian complex Lie algebras (vector spaces over $\mathbb{C}$)

$$X^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{C} \xrightarrow{\sim} \mathrm{Lie}\,\mathbf{T}_{\mathbb{C}}, \quad \chi \otimes v \mapsto d\chi(v), \quad \chi \in X^\vee,\ v \in \mathbb{C},$$
$$d\chi := d_1\chi\colon \mathbb{C} = \mathrm{Lie}\,\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Lie}\,\mathbf{T}_{\mathbb{C}}\,.$$

We obtain the standard embedding

$$X^\vee \hookrightarrow X^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{C} \xrightarrow{\sim} \operatorname{Lie} \mathbf{T}_\mathbb{C}\,, \quad \chi \mapsto \chi \otimes 1 \mapsto d\chi(1).$$

As usual, we set

$$P = \mathsf{X}(\mathbf{T}_\mathbb{C}^{\mathrm{sc}}), \quad Q = \mathsf{X}(\mathbf{T}_\mathbb{C}^{\mathrm{ad}});$$

these are the weight lattice and the root lattice. We set also

$$P^\vee = \mathsf{X}^\vee(\mathbf{T}_\mathbb{C}^{\mathrm{ad}}), \quad Q^\vee = \mathsf{X}^\vee(\mathbf{T}_\mathbb{C}^{\mathrm{sc}});$$

these are the coweight lattice and the coroot lattice. Then

$$Q \subset X \subset P \quad \text{and} \quad Q^\vee \subset X^\vee \subset P^\vee.$$

Let $\mathbf{G}$ and $\mathbf{T}$ be as above. We write $G = \mathbf{G}(\mathbb{R})$ for the set of $\mathbb{R}$-points of $\mathbf{G}$, and similarly we write $G^{\mathrm{ad}} = \mathbf{G}^{\mathrm{ad}}(\mathbb{R})$, $G^{\mathrm{sc}} = \mathbf{G}^{\mathrm{sc}}(\mathbb{R})$. We write $T = \mathbf{T}(\mathbb{R})$, and similarly we write $T^{\mathrm{ad}} = \mathbf{T}^{\mathrm{ad}}(\mathbb{R})$, $T^{\mathrm{sc}} = \mathbf{T}^{\mathrm{sc}}(\mathbb{R})$. We write $N = \mathbf{N}(\mathbb{R})$ and $W = \mathbf{W}(\mathbb{R})$. We write $Z_G = \mathbf{Z_G}(\mathbb{R})$ for the center of G.

We define an action of the group $X^\vee \rtimes W$ on the set $\mathfrak{t}$ as follows: an element $\chi \in X^\vee \subset \mathfrak{t}_\mathbb{C}$ acts by translation by $\boldsymbol{i}\chi \in \mathfrak{t}$ (where $\boldsymbol{i}^2 = -1$), and $w \in W \subset \operatorname{Aut} \mathbf{T}$ acts on $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$ as usual, i.e., as $d_1 w\colon \operatorname{Lie} \mathbf{T} \to \operatorname{Lie} \mathbf{T}$. It follows that the groups $Q^\vee \rtimes W$ and $P^\vee \rtimes W$ act on $\mathfrak{t}$.

Let $R = R(\mathbf{G}_\mathbb{C}, \mathbf{T}_\mathbb{C})$ denote the root system of $\mathbf{G}_\mathbb{C}$ with respect to $\mathbf{T}_\mathbb{C}$. Let $\Pi \subset R$ be a basis (a system of simple roots). Let $D = D(\mathbf{G}, \mathbf{T}, \Pi) = D(R, \Pi)$ denote the Dynkin diagram; the set of the vertices of D is Π.

Assume that $\mathbf{G}$ is (almost) simple. We write $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. Let $\widetilde{D} = \widetilde{D}(\mathbf{G}, \mathbf{T}, \Pi) = \widetilde{D}(R, \Pi)$ denote the extended Dynkin diagram; the set of vertices of $\widetilde{D}$ is $\widetilde{\Pi} = \{\alpha_1, \dots, \alpha_\ell, \alpha_0\}$, where $\alpha_1, \dots, \alpha_\ell$ are the simple roots, and α_0 is the lowest root. These roots $\alpha_1, \dots, \alpha_\ell, \alpha_0$ are linearly dependent, namely,

$$m_{\alpha_1}\alpha_1 + \dots + m_{\alpha_\ell}\alpha_\ell + m_{\alpha_0}\alpha_0 = 0, \tag{1}$$

where the coefficients m_{α_j} are positive integers for all $j = 1, \dots, \ell, 0$ and where $m_{\alpha_0} = 1$. We write m_j for m_{α_j}. These coefficients m_j are tabulated in [OV1, Table 6] and in [OV2, Table 3].

Now assume that $\mathbf{G}$ is semisimple, not necessarily simple. Then we have a decomposition $\mathbf{G} = \mathbf{G}^{(1)} \cdot \mathbf{G}^{(2)} \cdots \mathbf{G}^{(r)}$ into an almost direct product of simple groups. Then $\mathbf{T} = \mathbf{T}^{(1)} \cdot \mathbf{T}^{(2)} \cdots \mathbf{T}^{(r)}$ (an almost direct product of tori), where each $\mathbf{T}^{(k)}$ is a maximal torus in $\mathbf{G}^{(k)}$ $(k = 1, \dots, r)$. We write $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$, $\mathfrak{t}^{(k)} = \operatorname{Lie} \mathbf{T}^{(k)}$, then

$$\mathfrak{t} = \mathfrak{t}^{(1)} \oplus \dots \oplus \mathfrak{t}^{(r)}.$$

The root system R decomposes into a "direct sum" of irreducible root systems

$$R = R^{(1)} \sqcup \dots \sqcup R^{(r)}$$

(disjoint union), where $R^{(k)} = R(\mathbf{G}_\mathbb{C}^{(k)}, \mathbf{T}_\mathbb{C}^{(k)})$, and we have

$$\Pi = \Pi^{(1)} \sqcup \dots \sqcup \Pi^{(r)},$$

where each subset $\Pi^{(k)}$ $(k = 1, \dots, r)$ is a basis of $R^{(k)}$. We have

$$D = D^{(1)} \sqcup \dots \sqcup D^{(r)},$$

where each connected component $D^{(k)}$ $(k = 1, \dots, r)$ is the Dynkin diagram of the irreducible root system $R^{(k)}$ with respect to $\Pi^{(k)}$. Let $\alpha_0^{(k)} \in R^{(k)}$ denote the lowest root of $R^{(k)}$. Let $\widetilde{D}^{(k)}$ denote the extended Dynkin diagram of $R^{(k)}$ with respect to $\Pi^{(k)}$, then the set of vertices of $\widetilde{D}^{(k)}$ is $\widetilde{\Pi}^{(k)} := \Pi^{(k)} \cup \{\alpha_0^{(k)}\}$. We define the extended Dynkin diagram of R with respect to Π to be

$$\widetilde{D} = \widetilde{D}^{(1)} \sqcup \dots \sqcup \widetilde{D}^{(r)};$$

then the set of vertices of $\widetilde{D}$ is

$$\widetilde{\Pi} = \widetilde{\Pi}^{(1)} \sqcup \dots \sqcup \widetilde{\Pi}^{(r)} = \Pi \sqcup \widetilde{\Pi}_0,$$

where $\widetilde{\Pi}_0 = \{\alpha_0^{(1)}, \dots, \alpha_0^{(r)}\}$. For each $k = 1, \dots, r$, let $(m_\beta)_{\beta \in \widetilde{\Pi}^{(k)}}$ be the coefficients of linear dependence

$$\sum_{\beta \in \widetilde{\Pi}^{(k)}} m_\beta \beta = 0$$

normalized so that $m_{\alpha_0^{(k)}} = 1$. Then $m_\beta \in \mathbb{Z}, m_\beta \geq 0$ for any $\beta \in \widetilde{\Pi}$.

2. Action of $P^\vee/Q^\vee$ on the extended Dynkin diagram

First let $\mathbf{G}$ be a *simple* compact $\mathbb{R}$-group. Recall that $\mathfrak{t}$ denotes the Lie algebra of $\mathbf{T}$. Following [OV2, Section 3.3.6], we introduce the *barycentric coordinates* $x_{\alpha_1}, \dots, x_{\alpha_\ell}, x_{\alpha_0}$ of a point $x \in \mathfrak{t}$ by setting

$$d\alpha_j(x) = \boldsymbol{i} x_{\alpha_j} \quad \text{for } j = 1, \dots, \ell, \quad d\alpha_0(x) = \boldsymbol{i}(x_{\alpha_0} - 1),$$

where $\boldsymbol{i}^2 = -1$. We write x_j for x_{α_j}. By (1) we have

$$0 = \left(\sum_{j=0}^{\ell} m_j \, d\alpha_j \right)(x) = \boldsymbol{i} \left(-1 + \sum_{j=0}^{\ell} m_j x_j \right),$$

hence

$$\sum_{j=0}^{\ell} m_j x_j = 1. \tag{2}$$

By [Bou, Section VI.2.1] and [Bou, Section VI.2.2, Proposition 5(i)], see also [OV2, Section 3.3.6, Proposition 3.10(2)], the closed simplex $\Delta \subset \mathfrak{t}$ given by the inequalities

$$x_1 \geq 0, \ \dots, \ x_n \geq 0, \ x_0 \geq 0$$

is a fundamental domain for the affine Weyl group $Q^\vee \rtimes W$, where W is the usual Weyl group. This means that every orbit of $Q^\vee \rtimes W$ intersects Δ in one and only one point.

Now let $\mathbf{G}$ be a semisimple (not necessarily simple) compact $\mathbb{R}$-group. We introduce the barycentric coordinates $(x_\beta)_{\beta \in \widetilde{\Pi}}$ of x defined by

$$d\beta(x) = \boldsymbol{i} x_\beta \text{ for } \beta \in \Pi, \quad d\beta(x) = \boldsymbol{i}(x_\beta - 1) \text{ for } \beta \in \widetilde{\Pi}_0 = \widetilde{\Pi} \smallsetminus \Pi,$$

they satisfy

$$\sum_{\beta \in \widetilde{\Pi}^{(k)}} m_\beta x_\beta = 1 \quad \text{for each } k = 1, \dots, r,$$

see (2). Write $\mathfrak{t} = \bigoplus_{k=1}^{r} \mathfrak{t}_k$. For each $k = 1, \dots, r$, let $\Delta^{(k)}$ denote the closed simplex in $\mathfrak{t}^{(k)}$ given by the inequalities

$$x_\beta \geq 0 \quad \text{for } \beta \in \widetilde{\Pi}^{(k)}.$$

Then the product $\Delta = \prod_{k=1}^{r} \Delta^{(k)}$ is the closed subset in $\mathfrak{t}$ given by the inequalities

$$x_\beta \geq 0 \quad \text{for } \beta \in \widetilde{\Pi},$$

and Δ is a fundamental domain for the affine Weyl group $Q^\vee \rtimes W$ in $\mathfrak{t}$, acting as in Section 1. Again, this means that every orbit of $Q^\vee \rtimes W$ intersects Δ in one and only one point.

The group $(X^\vee \rtimes W)/(Q^\vee \rtimes W) = X^\vee/Q^\vee \simeq \pi_1(G)$ acts on Δ. We wish to describe this action. Since $X^\vee/Q^\vee \subset P^\vee/Q^\vee$, it suffices to describe the action of $P^\vee/Q^\vee$, and it suffices to consider the case when R is irreducible.

From now on till the end of this section we assume that R is an irreducible root system. The action of $P^\vee/Q^\vee$ on Δ is given by permutations of coordinates corresponding to a subgroup of the automorphism group of the extended Dynkin diagram acting simply transitively on the set of vertices α_j with $m_j = 1$. This action is described in [Bou, Section VI.2.3, Proposition 6].

Namely, let $\omega_1^\vee, \dots, \omega_\ell^\vee$ denote the set of fundamental coweights, i.e., the basis of $P^\vee$ dual to the basis $\alpha_1, \dots, \alpha_\ell$ of Q. Then the nonzero cosets of $P^\vee/Q^\vee$ are represented by the fundamental coweights $\omega_j^\vee$ such that $\boldsymbol{i}\omega_j^\vee$ belongs to Δ, i.e., by those $\omega_j^\vee$ with $m_j = 1$. Let w_0, resp. w_j, denote the longest element in W, resp. in the Weyl group W_j of the root subsystem R_j generated by $\Pi \smallsetminus \{\alpha_j\}$. Then the transformation

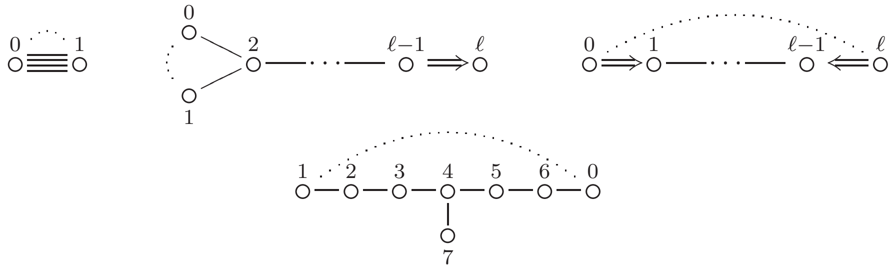
$$x \mapsto w_j w_0 x + \boldsymbol{i}\omega_j^\vee \tag{3}$$

preserves Δ whenever $m_j = 1$ and gives the action of the respective coset $[\omega_j^\vee] \in P^\vee/Q^\vee$ on Δ.

Observe that the affine transformation (3) is an isometry of the Euclidean structure on $\mathfrak{t}$ given by the restriction of the Killing form. Hence the action of $[\omega_j^\vee]$ preserves the Euclidean polytope structure of the simplex Δ. In particular, it permutes the vertices of Δ, which are equal to $v_i = \boldsymbol{i}\omega_i^\vee/m_i$ ($i = 1, \dots, \ell$) and $v_0 = 0$, and the facets Δ_i of Δ, which correspond to the roots $\alpha_i \in \widetilde{\Pi}$ ($i = 1, \dots, \ell, 0$), preserving the angles between the facets. Hence the action of $[\omega_j^\vee]$ induces a permutation $\sigma = \sigma_j$ of the set $\{1, \dots, \ell, 0\}$ such that the facet Δ_i maps to $\Delta_{\sigma(i)}$, and the opposite vertex v_i is mapped to $v_{\sigma(i)}$. In particular, σ_j takes 0 to j.

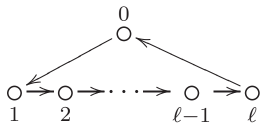
Since the relative lengths of the roots in $\widetilde{\Pi}$ and the angles between them and between the respective facets of Δ are read off from the extended Dynkin diagram $\widetilde{D}$, the permutation σ comes from an automorphism of $\widetilde{D}$. Furthermore, the action of $[\omega_j^\vee]$ permutes the barycentric coordinates x_i of a point $x \in \Delta$, because they are determined by the vertices $v_i \in \Delta$. Namely, any $x \in \Delta$ is mapped to $x' \in \Delta$ with coordinates $x'_i = x_{\sigma^{-1}(i)}$. One obtains an action of $P^\vee/Q^\vee$ on $\widetilde{D}$, which we describe below explicitly case by case, using [Bou, Planches I-IX, assertion (XII)].

If $\mathbf{G}$ is of one of the types $\mathbf{E}_8,\ \mathbf{F}_4, \mathbf{G}_2$, then $P^\vee/Q^\vee = 0$. If $\mathbf{G}$ is of one of the types $\mathbf{A}_1,\ \mathbf{B}_\ell\ (\ell \geq 3),\ \mathbf{C}_\ell\ (\ell \geq 2),\ \mathbf{E}_7$, then $P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$, and the nontrivial element $P^\vee/Q^\vee$ acts on $\widetilde{D}$ by the only nontrivial automorphism of $\widetilde{D}$:

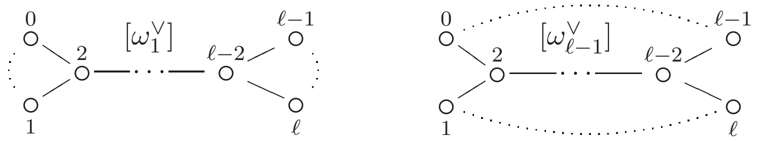


It remains to consider the cases $\mathbf{A}_\ell\ (\ell \geq 2)$, $\mathbf{D}_\ell$ and $\mathbf{E}_6$. In order to describe the action of the group $P^\vee/Q^\vee$ on $\widetilde{D}$, it suffices to describe its action on the set of vertices α_j of $\widetilde{D}$ with $m_j = 1$. These are the images of α_0 under the automorphism group of $\widetilde{D}$.

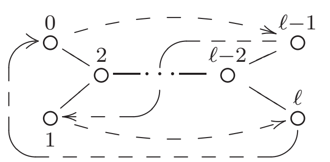
Let D be of type $\mathbf{A}_\ell$, $\ell \geq 2$. The generator $[\omega_1^\vee]$ of $P^\vee/Q^\vee$ acts on $\widetilde{D}$ as the cyclic permutation $0 \mapsto 1 \mapsto \ldots \mapsto \ell - 1 \mapsto \ell \mapsto 0$:



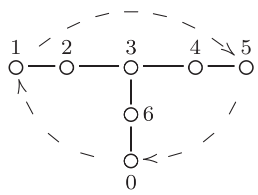
Let D be of type $\mathbf{D}_\ell$, $\ell \geq 4$ is even. We have $P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and the classes $[\omega_1^\vee]$ and $[\omega_{\ell-1}^\vee]$ are generators of $P^\vee/Q^\vee$. These generators act on $\widetilde{D}$ as follows: $[\omega_1^\vee]$ acts as $0 \leftrightarrow 1,\quad \ell - 1 \leftrightarrow \ell$, and $[\omega_{\ell-1}^\vee]$ acts as $0 \leftrightarrow \ell - 1,\quad 1 \leftrightarrow \ell$:



Let D be of type $\mathbf{D}_\ell$, $\ell \geq 5$ is odd. We have $P^\vee/Q^\vee \simeq \mathbb{Z}/4\mathbb{Z}$, and the class $[\omega_{\ell-1}^\vee]$ is a generator of $P^\vee/Q^\vee$. This generator acts on $\widetilde{D}$ as the 4-cycle $0 \mapsto \ell - 1 \mapsto 1 \mapsto \ell \mapsto 0$:



Let D be of type $\mathbf{E}_6$. The generator $[\omega_1^\vee] \in P^\vee/Q^\vee$ acts as the 3-cycle $0 \mapsto 1 \mapsto 5 \mapsto 0$:



3. n-th roots of a central element

Let $\mathbf{G}$ a compact semisimple $\mathbb{R}$-group, not necessarily simple. Let $\mathbf{T}$, G, T, X, D, $\widetilde{D}$, ets. be as in Section 1.

Let $z \in Z_G$ and let n be a positive integer. We consider the set of n-th roots of z in G

$$G_n^z := \{g \in G \mid g^n = z\}.$$

In particular, $G_n := G_n^1$ is the set of n-th roots of 1 in G, i.e., the set of elements of order dividing n in G.

The group G acts on G_n^z on the left by conjugation $g * a = gag^{-1}$ $(g \in G,\ a \in G_n^z)$. We wish to compute the set $G_n^z/\!\sim$ of n-th roots of z modulo conjugation.

Consider the set $T_n^z \subset G_n^z$ (note that $z \in Z \subset T$). The group W acts on T_n^z on the left by

$$w * t = ntn^{-1}, \tag{4}$$

where $w = nT \in W,\ n \in N,\ t \in T$. It is easy to see that the embedding $T_n^z \hookrightarrow G_n^z$ induces a bijection $T_n^z/W \xrightarrow{\sim} G_n^z/\!\sim$. Thus we wish to compute T_n^z/W.

We describe the set T_n^z/W in terms of *Kac n-labelings of $\widetilde{D}$*.

Definition 3.1. A *Kac n-labeling* of an extended Dynkin diagram $\widetilde{D} = \widetilde{D}^{(1)} \sqcup \dots \sqcup \widetilde{D}^{(r)}$, where each $\widetilde{D}^{(k)}$ is connected for $k = 1, \dots, r$, is a family of nonnegative integer numerical labels $\boldsymbol{p} = (p_\beta)_{\beta \in \widetilde{\Pi}} \in \mathbb{Z}_{\ge 0}^{\widetilde{\Pi}}$ at the vertices $\beta \in \widetilde{\Pi}$ of $\widetilde{D}$ satisfying

$$\sum_{\beta \in \widetilde{\Pi}^{(k)}} m_\beta p_\beta = n \quad \text{for each } k = 1, \dots, r. \tag{5}$$

Note that a Kac n-labeling $\boldsymbol{p}$ of $\widetilde{D} = \widetilde{D}^{(1)} \sqcup \dots \sqcup \widetilde{D}^{(r)}$ is the same as a family $(\boldsymbol{p}^{(1)}, \dots \boldsymbol{p}^{(r)})$, where each $\boldsymbol{p}^{(k)}$ is a Kac n-labeling of $\widetilde{D}^{(k)}$.

Let $z \in Z_G \subset T$. We write

$$z = \exp 2\pi \boldsymbol{i} \zeta, \quad \text{where} \ \ \zeta \in \mathfrak{t}_\mathbb{C}. \tag{6}$$

For $\lambda \in X$ consider $d\lambda(\zeta) \in \mathbb{C}$. We have

$$\exp 2\pi \boldsymbol{i}\, d\lambda(\zeta) = \exp d\lambda(2\pi \boldsymbol{i} \zeta) = \lambda(\exp 2\pi \boldsymbol{i} \zeta) = \lambda(z). \tag{7}$$

Since z is an element of finite order in T, we see that $\lambda(z)$ is a root of unity, hence by (7) $d\lambda(\zeta) \in \mathbb{Q}$, and it follows from (7) that the image of $d\lambda(\zeta)$ in $\mathbb{Q}/\mathbb{Z}$ depends only on z, and not on the choice of ζ. Note that if $\lambda \in Q \subset X$, then $\lambda(z) = 1$, hence $d\lambda(\zeta) \in \mathbb{Z}$.

Notation 3.2. We denote by $\mathcal{K}_n$ the set of Kac n-labelings of $\widetilde{D}$, i.e., the set of $\boldsymbol{p} = (p_\beta) \in \mathbb{Z}_{\geq 0}^{\widetilde{\Pi}}$ satisfying (5). We denote by $\mathcal{K}_{n,\mathbb{R}}$ the set of families $\boldsymbol{p} = (p_\beta) \in \mathbb{R}_{\geq 0}^{\widetilde{\Pi}}$ satisfying (5), i.e., the set of tuples of barycentric coordinates of points in $n\Delta$. For $z \in Z_G$, we denote by $\mathcal{K}_n^z$ the set of Kac n-labelings $\boldsymbol{p} \in \mathcal{K}_n$ of $\widetilde{D}$ satisfying

$$\sum_{\alpha\in\Pi} c_\alpha p_\alpha \equiv d\lambda(\zeta) \pmod{\mathbb{Z}} \tag{8}$$

$$\text{for any generator } [\lambda] \text{ of } X/Q \text{ with } \lambda = \sum_{\alpha\in\Pi} c_\alpha \alpha\,,$$

where ζ is as in (6). Condition (8) does not depend on the choice of ζ satisfying (6). We have $\mathcal{K}_n^z \subset \mathcal{K}_n \subset \mathcal{K}_{n,\mathbb{R}}$. The group $X^\vee/Q^\vee$ acts on $\mathcal{K}_{n,\mathbb{R}}$ and $\mathcal{K}_n$ via the action on $\widetilde{D}$. We shall see below that the subset $\mathcal{K}_n^z$ of $\mathcal{K}_n$ is $X^\vee/Q^\vee$-invariant.

Construction 3.3. Let $\boldsymbol{p} = (p_\beta) \in \mathcal{K}_{n,\mathbb{R}}$. Set

$$\mathbf{x} = (x_\beta)_{\beta\in\widetilde{\Pi}} := (p_\beta/n)_{\beta\in\widetilde{\Pi}} \in \mathcal{K}_{1,\mathbb{R}},$$

then there exists a point $x \in \Delta \subset \mathfrak{t}$ with barycentric coordinates $(x_\beta)_{\beta\in\widetilde{\Pi}}$. We set

$$\varphi(\boldsymbol{p}) = e(x) := \exp 2\pi x \in T.$$

The following theorem gives a combinatorial description of the set T_n^z/W in terms of Kac n-labelings. It generalizes a result of Kac [K], who described, in particular, the set T_n/W in the case when $\mathbf{G}$ is an adjoint group.

Theorem 3.4. *Let $\mathbf{G}$ be a compact semisimple $\mathbb{R}$-group, $\mathbf{T} \subset \mathbf{G}$ be a maximal torus, $R = R(\mathbf{G}_\mathbb{C}, \mathbf{T}_C)$ be the corresponding root system, Π be a basis of R, $\widetilde{D} = \widetilde{D}(\mathbf{G}, \mathbf{T}, \Pi)$ be the corresponding extended Dynkin diagram. Let n be a positive integer. Let $z \in Z_G$ be a central element. Then the subset $\mathcal{K}_n^z \subset \mathcal{K}_n$ is $X^\vee/Q^\vee$-invariant, and the map $\varphi\colon \mathcal{K}_{n,\mathbb{R}} \to T$ of Construction 3.3 induces a bijection*

$$\varphi_*\colon \mathcal{K}_n^z/(X^\vee/Q^\vee) \xrightarrow{\sim} T_n^z/W \tag{9}$$

between the set of $X^\vee/Q^\vee$-orbits in $\mathcal{K}_n^z$ and the set of W-orbits in T_n^z.

Proof. Consider a W-orbit $[a]$ in T/W, where $a \in T$. Write $a = e(x)$ for some $x \in \mathfrak{t}$. The map $e\colon \mathfrak{t} \to T$ is W-equivariant. The group $X^\vee$ acts on the set $\mathfrak{t}$ by translations, and the map e induces a bijection $\mathfrak{t}/X^\vee \xrightarrow{\sim} T$, hence it induces a bijection

$$\mathfrak{t}/(X^\vee \rtimes W) \xrightarrow{\sim} T/W.$$

Since Δ is a fundamental domain of the normal subgroup $Q^\vee \rtimes W \subset X^\vee \rtimes W$ (see Section 2), after changing the representative $a \in T$ of $[a] \in T/W$ we may choose x lying in Δ, and such x is unique up to the action of the quotient group $(X^\vee \rtimes W)/(Q^\vee \rtimes W) = X^\vee/Q^\vee$. We see that the map e induces a bijection

$$\Delta/(X^\vee/Q^\vee) \xrightarrow{\sim} T/W.$$

The map

$$\mathcal{K}_{n,\mathbb{R}} \to \Delta, \quad \boldsymbol{p} \mapsto \mathbf{x} = \boldsymbol{p}/n \mapsto x \tag{10}$$

is a $P^\vee/Q^\vee$-equivariant bijection, hence it induces a bijection

$$\mathcal{K}_{n,\mathbb{R}}/(X^\vee/Q^\vee) \xrightarrow{\sim} \Delta/(X^\vee/Q^\vee).$$

We see that the map $\varphi\colon \mathcal{K}_{n,\mathbb{R}} \to T$ induces a bijection

$$\mathcal{K}_{n,\mathbb{R}}/(X^\vee/Q^\vee) \xrightarrow{\sim} T/W. \tag{11}$$

In particular, two tuples $\boldsymbol{p}, \boldsymbol{p'} \in \mathcal{K}_{n,\mathbb{R}}$ are in the same $X^\vee/Q^\vee$-orbit if and only if $\varphi(\boldsymbol{p}), \varphi(\boldsymbol{p'}) \in T$ are in the same W-orbit.

Now we wish to describe $\boldsymbol{p} = (p_\beta) \in \mathcal{K}_{n,\mathbb{R}}$ such that $\varphi(\boldsymbol{p}) \in T_n^z$, i.e., $\varphi(\boldsymbol{p})^n = z$. For $x \in \Delta$ obtained from $\boldsymbol{p} \in \mathcal{K}_{n,\mathbb{R}}$ as in (10), the assertion that $e(x)^n = z$ is equivalent to the condition

$$\lambda(\exp 2\pi n x) = \lambda(\exp 2\pi \boldsymbol{i} \zeta)$$

for all $\lambda \in X$, which in turn is equivalent to

$$-\boldsymbol{i} n\, d\lambda(x) \equiv d\lambda(\zeta) \pmod{\mathbb{Z}}.$$

We write $\lambda = \sum_{\alpha\in\Pi} c_\alpha \alpha$ and obtain

$$-\boldsymbol{i} n \sum_{\alpha\in\Pi} c_\alpha\, d\alpha(x) \equiv d\lambda(\zeta) \pmod{\mathbb{Z}}.$$

Since $d\alpha(x) = \boldsymbol{i} x_\alpha$ for $\alpha \in \Pi$, and $n x_\alpha = p_\alpha$, we obtain

$$\sum_{\alpha\in\Pi} c_\alpha p_\alpha = n \sum_{\alpha\in\Pi} c_\alpha x_\alpha \equiv d\lambda(\zeta) \pmod{\mathbb{Z}}.$$

Thus $\varphi(\boldsymbol{p}) \in T_n^z$ if and only if

$$\sum_{\alpha\in\Pi} c_\alpha p_\alpha \equiv d\lambda(\zeta) \pmod{\mathbb{Z}} \ \text{ for any } \ \lambda \in X \text{ with } \ \lambda = \sum_{\alpha\in\Pi} c_\alpha \alpha. \tag{12}$$

Assume that $\varphi(\boldsymbol{p}) \in T_n^z$, then (12) holds. Observe that for $\lambda = \alpha \in \Pi$, condition (12) means that $p_\alpha \in \mathbb{Z}$, because $d\alpha(\zeta) \in \mathbb{Z}$. Since $p_\alpha \in \mathbb{Z}$ for all $\alpha \in \Pi$, by (5) we have $p_\beta \in \mathbb{Z}$ for any $\beta \in \widetilde{\Pi}_0 = \widetilde{\Pi} \smallsetminus \Pi$, because $m_\beta = 1$. Thus $\boldsymbol{p} \in \mathcal{K}_n$. Condition (8) is a special case of (12). We conclude that $\boldsymbol{p} \in \mathcal{K}_n^z$.

Conversely, assume that $\boldsymbol{p} \in \mathcal{K}_n^z \subset \mathcal{K}_n$, then condition (12) holds for $\lambda = \alpha$ for any $\alpha \in \Pi$. Since condition (12) is additive in λ (i.e., it holds for any integer linear combination of two weights $\lambda, \lambda' \in P$ whenever it holds for λ and λ'), it holds for any $\lambda \in Q$, because Π generates Q as an abelian group. Now condition (8) implies that (12) holds for for all $\lambda \in X$. We conclude that $\varphi(\boldsymbol{p}) \in T_n^z$.

Thus $\varphi(\boldsymbol{p}) \in T_n^z$ if and only if $\boldsymbol{p} \in \mathcal{K}_n^z$. Since the subset $T_n^z \subset T$ is W-invariant, we conclude that the subset $\mathcal{K}_n^z \subset K_{n,\mathbb{R}}$ is $X^\vee/Q^\vee$-invariant. Bijection (11) induces (9), which proves the theorem. $\square$

We need another version of Theorem 3.4. We start from a Kac n-labeling $\boldsymbol{q} = (q_\beta) \in \mathcal{K}_n$ of $\widetilde{D}$. Set $z = \varphi(\boldsymbol{q})^n$. It follows from the proof of Theorem 3.4 that $z \in Z_G$.

Corollary 3.5. *With the assumptions and notation of Theorem 3.4, let* $\boldsymbol{q}$ *be an* n*-labeling of* $\widetilde{D}$. *Set* $z = \varphi(\boldsymbol{q})^n \in Z_G$. *Then the subset* $\mathcal{K}_n^{(\boldsymbol{q})} \subset \mathcal{K}_n$ *consisting of Kac* n*-labelings* $\boldsymbol{p} \in \mathcal{K}_n$ *of* $\widetilde{D}$ *satisfying*

$$\sum_{\alpha\in\Pi} c_\alpha p_\alpha \equiv \sum_{\alpha\in\Pi} c_\alpha q_\alpha \pmod{\mathbb{Z}} \tag{13}$$

$$\text{for any generator } [\lambda] \text{ of } X/Q \text{ with } \lambda = \sum_{\alpha\in\Pi} c_\alpha \alpha\,,$$

is $X^\vee/Q^\vee$*-invariant, and the map* φ *of Construction 3.3 induces a bijection between* $\mathcal{K}_n^{(\boldsymbol{q})}/(X^\vee/Q^\vee)$ *and* T_n^z/W.

Indeed, by Theorem 3.4 we have $\boldsymbol{q} \in \mathcal{K}_n^z$, hence $\mathcal{K}_n^{(\boldsymbol{q})} = \mathcal{K}_n^z$, and the corollary follows from the theorem.

4. Real Galois cohomology

We denote by $H^1(\mathbb{R},\mathbf{H})$ the first (nonabelian) Galois cohomology set of an $\mathbb{R}$-group $\mathbf{H}$. By definition, $H^1(\mathbb{R},\mathbf{H}) = Z^1(\mathbb{R},\mathbf{H})/\!\sim$, where $Z^1(\mathbb{R},\mathbf{H}) = \{c \in \mathbf{H}(\mathbb{C}) \mid c\bar{c} = 1\}$, and $c \sim c'$ if there exists $h \in \mathbf{H}(\mathbb{C})$ such that $c' = h^{-1}c\bar{h}$. We say that $c \in Z^1(\mathbb{R},\mathbf{H})$ is a *cocycle.*

Let $\mathbf{H}(\mathbb{R})_2 \subset \mathbf{H}(\mathbb{R})$ denote the subset of elements of order dividing 2. If $b \in \mathbf{H}(\mathbb{R})_2$, then

$$b\bar{b} = b^2 = 1,$$

hence b is a cocycle. Thus $\mathbf{H}(\mathbb{R})_2 \subset Z^1(\mathbb{R},\mathbf{H})$.

Let $\mathbf{G}$ be a connected, compact (anisotropic), semisimple algebraic group over the field of real numbers $\mathbb{R}$. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. We use the notation of Section 1.

Theorem 4.1. *Let* $\mathbf{G}$ *be a connected, compact, semisimple algebraic* $\mathbb{R}$*-group. There is a canonical bijection between the set of* $P^\vee/Q^\vee$*-orbits in the set* $\mathcal{K}_2$ *of Kac* 2*-labelings of the extended Dynkin diagram* $\widetilde{D} = \widetilde{D}(\mathbf{G},\mathbf{T},\Pi)$ *and the first Galois cohomology set* $H^1(\mathbb{R},\mathbf{G}^{\mathrm{ad}})$.

We specify the bijection. Consider the map $\varphi^{\mathrm{ad}}\colon \mathcal{K}_{2,\mathbb{R}} \to T^{\mathrm{ad}}$ of Construction 3.3 for $\mathbf{G}^{\mathrm{ad}}$, it sends $\mathcal{K}_2 \subset \mathcal{K}_{2,\mathbb{R}}$ to $(T^{\mathrm{ad}})_2$, where $(T^{\mathrm{ad}})_2$ denotes the set of elements of order dividing 2 in T^{ad}. The bijection of the theorem sends the $P^\vee/Q^\vee$-orbit of $\boldsymbol{p} \in \mathcal{K}_2$ to the cohomology class $[\varphi(\boldsymbol{p})] \in H^1(\mathbb{R},\mathbf{G}^{\mathrm{ad}})$ of $\varphi(\boldsymbol{p}) \in (T^{\mathrm{ad}})_2 \subset Z^1(\mathbb{R},\mathbf{G}^{\mathrm{ad}})$.

This result goes back to Kac [K]. In the last sentence of [K] Kac notes that his results yield a classification of real forms of simple Lie algebras. Inner real forms of a compact simple group $\mathbf{G}$ (or of its Lie algebra Lie $\mathbf{G}$) are classified by the orbits of the group $\operatorname{Aut}\widetilde{D} = (P^\vee/Q^\vee) \rtimes \operatorname{Aut} D$ in the set $\mathcal{K}_2$ of Kac 2-labelings of $\widetilde{D}$. Those orbits and the corresponding real forms are listed in [OV1, Table 7, Types I and II].

Proof. By Theorem 3.4 for the adjoint group $\mathbf{G}^{\mathrm{ad}}$, the map φ^{ad} induces a bijection $\mathcal{K}_2/(P^\vee/Q^\vee) \xrightarrow{\sim} (T^{\mathrm{ad}})_2/W$. By [S, Section III.4.5, Example (a)] the map sending an element $t^{\mathrm{ad}} \in (T^{\mathrm{ad}})_2 \subset Z^1(\mathbb{R},\mathbf{G}^{\mathrm{ad}})$ to its cohomology

class $[t^{\mathrm{ad}}] \in H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$ induces a bijection $(T^{\mathrm{ad}})_2/W \xrightarrow{\sim} H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$, and the theorem follows. □

Let ${}_c\mathbf{G}$ be an inner twisted form of a compact semisimple $\mathbb{R}$-group $\mathbf{G}$, where $c \in Z^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$. By Theorem 4.1 the cocycle c is equivalent to a cocycle of the form $t^{\mathrm{ad}} = \varphi^{\mathrm{ad}}(\boldsymbol{q}) \in (T^{\mathrm{ad}})_2 \subset Z^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$ for some Kac 2-labeling $\boldsymbol{q} = (q_\beta)_{\beta\in\widetilde{\Pi}}$ of $\widetilde{D}$. We have $t^{\mathrm{ad}} = \exp 2\pi y$, where $y \in \Delta$ has barycentric coordinates $y_\beta = q_\beta/2$ for $\beta \in \widetilde{\Pi}$. It follows that t^{ad} is determined by the equations

$$\alpha(t^{\mathrm{ad}}) = (-1)^{q_\alpha} \quad \text{for } \alpha \in \Pi.$$

We can twist $\mathbf{G}$ using t^{ad}; we denote the obtained twisted form by ${}_{\boldsymbol{q}}\mathbf{G}$, then ${}_c\mathbf{G} \simeq {}_{\boldsymbol{q}}\mathbf{G}$. Note that there is a canonical isomorphism between $\mathbf{T}$ and the twisted torus ${}_{\boldsymbol{q}}\mathbf{T}$, because the inner automorphism of $\mathbf{G}$ defined by t^{ad} acts on $\mathbf{T}$ trivially. It follows that $\mathbf{T}$ canonically embeds into ${}_{\boldsymbol{q}}\mathbf{G}$, in particular, $T_2 \subset {}_{\boldsymbol{q}}\mathbf{G}(\mathbb{R})_2 \subset Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$.

We compute $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$. Set $t = \varphi(\boldsymbol{q}) \in T$, where $\varphi\colon \mathcal{K}_{2,\mathbb{R}} \to T$ is the map of Construction 3.3. Then the image of t in T^{ad} is t^{ad}. Since $(t^{\mathrm{ad}})^2 = 1$, we see that $t^2 \in Z_G$. Set $z = t^2$, then $t \in T_2^z$.

Lemma 4.2. *There is a bijection $T_2^z/W \xrightarrow{\sim} H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ that sends the W-orbit of $a \in T_2^z$ to the cohomology class of $at^{-1} \in T_2 \subset Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$.*

Proof. Recall that we have the standard left action $*$ of W on T_2^z given by formula (4). We define the t^{ad}-twisted left action $*_{t^{\mathrm{ad}}}$ of W on T_2 as follows: let $w = nT \in W$, $n \in N$, $b \in T_2$, then

$$w *_{t^{\mathrm{ad}}} b = nbtn^{-1}t^{-1}.$$

We define a bijection

$$a \mapsto at^{-1}\colon T_2^z \to T_2 \tag{14}$$

(which takes t to 1). We have

$$(w * a)t^{-1} = nan^{-1}t^{-1} = n(at^{-1})tn^{-1}t^{-1} = w *_{t^{\mathrm{ad}}} (at^{-1}),$$

hence, the standard left action $*$ of W on T_2^z is compatible with the t^{ad}-twisted left action $*_{t^{\mathrm{ad}}}$ of W on T_2 with respect to bijection (14). We obtain a bijection $T_2^z/W = T_2^z/ * W \xrightarrow{\sim} T_2/ *_{t^{\mathrm{ad}}} W$ between the sets of W-orbits.

By [B1, Theorem 1], see also [B2, Theorem 9], the map sending $b \in T_2 \subset Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ to its cohomology class $[b] \in H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ induces a bijection $T_2/ *_{t^{\mathrm{ad}}} W \xrightarrow{\sim} H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$.

Combining these two bijections, we obtain the bijection of the lemma. □

The following theorem is the main result of this paper. It gives a combinatorial description of the first Galois cohomology set $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ of an inner twisted form ${}_{\boldsymbol{q}}\mathbf{G}$ of a compact semisimple $\mathbb{R}$-group $\mathbf{G}$ in terms of Kac 2-labelings of the extended Dynkin diagram of $\mathbf{G}$.

Theorem 4.3. *Let $\mathbf{G}$ be a connected, compact, semisimple algebraic $\mathbb{R}$-group. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus and Π be a basis of the root system $R = R(\mathbf{G}_\mathbb{C}, \mathbf{T}_\mathbb{C})$. Let $\boldsymbol{q}$ be a Kac 2-labeling of the extended Dynkin diagram $\widetilde{D} = \widetilde{D}(\mathbf{G}, \mathbf{T}, \Pi)$. Then the subset $\mathcal{K}_2^{(\boldsymbol{q})} \subset \mathcal{K}_2$ of Kac 2-labelings $\boldsymbol{p}$ of $\widetilde{D}$*

satisfying condition (13) *of Corollary 3.5 is* $X^\vee/Q^\vee$*-invariant, and there is a bijection between the set of orbits* $\mathcal{K}_2^{(\boldsymbol{q})}/(X^\vee/Q^\vee)$ *and the first Galois cohomology set* $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$.

We specify the bijection of the theorem. It is induced by the map sending a Kac 2-labeling $\boldsymbol{p} \in \mathcal{K}_2$ satisfying (13) to the cocycle $\exp 2\pi u \in T_2 \subset Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$, where $u \in \mathfrak{t}$ is the the element with barycentric coordinates $u_\alpha = (p_\alpha - q_\alpha)/2$ for $\alpha \in \Pi$. In particular, this bijection sends the $X^\vee/Q^\vee$-orbit of $\boldsymbol{q}$ to the neutral element of $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$.

Proof of Theorem 4.3. By Corollary 3.5 there is a bijection between the set of orbits of $X^\vee/Q^\vee$ in the set of Kac 2-labelings $\boldsymbol{p} \in \mathcal{K}_2$ of $\widetilde{D}$ satisfying (13) and the set T_2^z/W, which sends the $X^\vee/Q^\vee$-orbit of $\boldsymbol{p}$ to the W-orbit of $\exp 2\pi x \in T_2^z$, where $x \in \mathfrak{t}$ is the element with barycentric coordinates $x_\beta = p_\beta/2$ for $\beta \in \widetilde{\Pi}$. By Lemma 4.2 there is a bijection $T_2^z/W \xrightarrow{\sim} H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$, which sends the W-orbit of an element $a \in T_2^z$ to the cohomology class of $at^{-1} \in T_2 \subset Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$. We compose these two bijections. Since $t = \exp 2\pi y$, where $y \in \mathfrak{t}$ is the element with barycentric coordinates $y_\beta = q_\beta/2$ for $\beta \in \widetilde{\Pi}$, the composite bijection sends the $X^\vee/Q^\vee$-orbit of a Kac 2-labeling $\boldsymbol{p}$ satisfying (13) to the cohomology class of

$$\exp 2\pi x \cdot (\exp 2\pi y)^{-1} = \exp 2\pi(x-y) = \exp 2\pi u \in T_2 \subset Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G}),$$

where $u := x - y \in \mathfrak{t}$ has barycentric coordinates $u_\alpha = (p_\alpha - q_\alpha)/2$ for $\alpha \in \Pi$. Clearly this composite bijection sends $\boldsymbol{p} = \boldsymbol{q}$ to the cohomology class of $1 \in Z^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$, thus to the neutral element of $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$. □

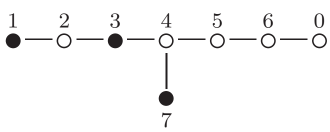
5. Example: forms of $\mathbf{E}_7$

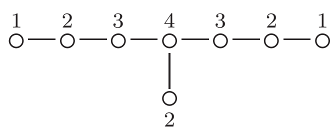
Let $\mathbf{G}$ be the simply connected compact group $\mathbf{G}$ of type $\mathbf{E}_7$. Since $\mathbf{G}$ is simply connected, we have $X = P$.

Below in the left hand side we give the extended Dynkin diagram $\widetilde{D}$ of $\mathbf{G}_\mathbb{C}$ with the numbering of vertices of [OV1, Table 1], and in the right hand side we give $\widetilde{D}$ with the coefficients m_j from [OV1, Table 6], see (1). We have $X/Q = P/Q \simeq \mathbb{Z}/2\mathbb{Z}$, and there is $\lambda \in X \smallsetminus Q$ with

$$\lambda = \tfrac{1}{2}(\alpha_1 + \alpha_3 + \alpha_7), \tag{15}$$

see e.g. [OV1, Table 3]. In the left-hand side diagram below we mark in black the roots appearing (with non-integer half-integer coefficients) in formula (15):





The Kac 2-labelings of $\widetilde{D}$ are:

$$\boldsymbol{q}^{(1)} = \begin{matrix}0000002\\ 0\end{matrix} \qquad \boldsymbol{q}^{(2)} = \begin{matrix}2000000\\ 0\end{matrix}$$

$$\boldsymbol{q}^{(3)} = \begin{matrix}1000001\\ 0\end{matrix}$$

$$\boldsymbol{q}^{(4)} = \begin{matrix}0100000\\ 0\end{matrix} \qquad \boldsymbol{q}^{(5)} = \begin{matrix}0000010\\ 0\end{matrix}$$

$$\boldsymbol{q}^{(6)} = \begin{matrix}0000000\\ 1\end{matrix}.$$

The real forms of $\mathbf{E}_7$ correspond to elements of $H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$, and by Theorem 4.1 to the orbits of $P^\vee/Q^\vee$ in the set $\mathcal{K}_2$ of Kac 2-labelings of $\widetilde{D}$. These orbits are:

$$\{\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}\}, \quad \{\boldsymbol{q}^{(3)}\}, \quad \{\boldsymbol{q}^{(4)}, \boldsymbol{q}^{(5)}\}, \quad \{\boldsymbol{q}^{(6)}\},$$

hence $\# H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}}) = 4$.

Concerning $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$, condition (13) defining $\mathcal{K}_2^{(\boldsymbol{q})}$ reads

$$\tfrac{1}{2}(p_1 + p_3 + p_7) \equiv \tfrac{1}{2}(q_1 + q_3 + q_7) \pmod{\mathbb{Z}},$$

which is equivalent to

$$p_1 + p_3 + p_7 \equiv q_1 + q_3 + q_7 \pmod{2}.$$

We say that a 2-labeling $\boldsymbol{p} \in \mathcal{K}_2$ is *even* (resp., *odd*) if the sum over the black vertices

$$p_1 + p_3 + p_7$$

is even (resp., odd). Then $\mathcal{K}_2^{(\boldsymbol{q})}$ is the set of labelings $\boldsymbol{p} \in \mathcal{K}_2$ of the same parity as $\boldsymbol{q}$. Since $\mathbf{G}$ is simply connected, we have $X^\vee = Q^\vee$, and by Theorem 4.3 the first Galois cohomology set $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ is in a bijection with the set $\mathcal{K}_2^{(\boldsymbol{q})}$.

For ${}_{\boldsymbol{q}}\mathbf{G} = \mathbf{E}_7$ (the compact form) we take $\boldsymbol{q} = \boldsymbol{q}^{(1)}$, then $q_1 + q_3 + q_7 = 0$, hence $\boldsymbol{q}$ is even. For ${}_{\boldsymbol{q}}\mathbf{G} = \mathbf{EVI}$ we take $\boldsymbol{q} = \boldsymbol{q}^{(4)}$, see [OV1, Table 7]. We have $q_1 + q_3 + q_7 = 0$, so again $\boldsymbol{q}$ is even. We see that in both cases the set $\mathcal{K}_2^{(\boldsymbol{q})}$ is the set of all *even* 2-labelings of $\widetilde{D}$:

$$\begin{matrix}0000002\\ 0\end{matrix} \qquad \begin{matrix}2000000\\ 0\end{matrix} \qquad \begin{matrix}0100000\\ 0\end{matrix} \qquad \begin{matrix}0000010\\ 0\end{matrix}. \tag{16}$$

The set $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ is in a bijection with the set (16). In particular, $\# H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G}) = 4$ in both the compact case and **EVI**.

For ${}_{\boldsymbol{q}}\mathbf{G} = \mathbf{EV}$ (the split form) we take $\boldsymbol{q} = \boldsymbol{q}^{(6)}$, see [OV1, Table 7]. We have $q_1 + q_3 + q_7 = 1$, hence $\boldsymbol{q}$ is odd. For ${}_{\boldsymbol{q}}\mathbf{G} = \mathbf{EVII}$ (the Hermitian form) we take $\boldsymbol{q} = \boldsymbol{q}^{(3)}$, see [OV1, Table 7]. Again we have $q_1 + q_3 + q_7 = 1$, and again $\boldsymbol{q}$ is odd. In both cases the set $\mathcal{K}_2^{(\boldsymbol{q})}$ is the set of all *odd* 2-labelings of $\widetilde{D}$:

$$\begin{matrix}1000001\\ 0\end{matrix} \qquad \begin{matrix}0000000\\ 1\end{matrix}. \tag{17}$$

The set $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$ is in a bijection with the set (17). In particular, $\# H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G}) = 2$ in both cases **EV** and **EVII**.

In each case the element $\boldsymbol{q} \in \mathcal{K}_2^{(q)}$ corresponds to the neutral element of $H^1(\mathbb{R}, {}_{\boldsymbol{q}}\mathbf{G})$.

6. Example: half-spin groups

Let $\mathbf{G}$ be the compact group of type $\mathbf{D}_\ell$ with even $\ell = 2k \geq 4$ with the cocharacter lattice

$$X^\vee = \langle Q^\vee, \omega_{\ell-1}^\vee \rangle.$$

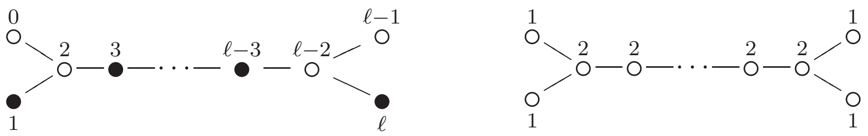
This compact group is neither simply connected nor adjoint, and it is isomorphic to $\mathbf{SO}_{2\ell}$ only if $\ell = 4$. It is called a half-spin group.

We show that the character lattice X is generated by Q and the weight

$$\lambda := (\alpha_1 + \alpha_3 + \cdots + \alpha_{\ell-3} + \alpha_\ell)/2. \tag{18}$$

Indeed, λ is orthogonal to $\omega_{\ell-1}^\vee$ and $\langle \lambda, \alpha^\vee \rangle = 0, 1, -1 \in \mathbb{Z}$ for any $\alpha \in \Pi$. We see that $\lambda \in X$. Since $\lambda \notin Q$ and $[X : Q] = 2$, we conclude that $X = \langle Q, \lambda \rangle$.

Below in the left hand side we give the extended Dynkin diagram $\widetilde{D}$ of $\mathbf{G}_\mathbb{C}$ with the numbering of vertices of [OV1, Table 1] (which coincides with the labeling of Bourbaki [Bou]). We mark in black the roots that appear (with non-integer half-integer coefficients) in the formula (18) for λ. In the right hand side we give $\widetilde{D}$ with the coefficients m_j from [OV1, Table 6], see (1):



Let $\boldsymbol{p}$ be a Kac 2-labeling of the extended Dynkin diagram $\widetilde{D}$. We say that $\boldsymbol{p}$ is *even* (resp., *odd*), if the sum over the black vertices

$$p_1 + p_3 + \cdots + p_{\ell-3} + p_\ell$$

is even (resp., odd). If $\boldsymbol{q} \in \mathcal{K}_2$ is a Kac 2-labeling of $\widetilde{D}$, then $K_2^{(\boldsymbol{q})}$ is the set of Kac 2-labelings $\boldsymbol{p}$ of the same parity as $\boldsymbol{q}$.

The group $X^\vee/Q^\vee = \{0, [\omega_{\ell-1}^\vee]\}$ acts on $\widetilde{D}$ and on the set $\mathcal{K}_2$ of Kac 2-labelings of $\widetilde{D}$. The nontrivial element $\sigma := [\omega_{\ell-1}^\vee] \in X^\vee/Q^\vee$ acts as the reflection with respect to the vertical axis of symmetry of $\widetilde{D}$, see Section 2, and clearly preserves the parity of labelings. We say that a σ-orbit in $\mathcal{K}_2$ is even (resp., odd), if it consists of even (resp., odd) 2-labelings.

Let $\boldsymbol{q}$ be a 2-labeling of $\widetilde{D}$. By Theorem 4.3 the cohomology set $H^1(\mathbb{R}, {}_{\boldsymbol{q}}G)$ is in a bijection with the set $\mathcal{K}_2^{(\boldsymbol{q})}/(X^\vee/Q^\vee)$, i.e., with the set of σ-orbits in $\mathcal{K}_2$ of the same parity as $\boldsymbol{q}$. Thus in order to compute $H^1(\mathbb{R}, {}_{\boldsymbol{q}}G)$ for all 2-labelings $\boldsymbol{q}$ of $\widetilde{D}$, it suffices to compute the sets $\mathrm{Orb}^{\mathrm{even}}(\mathbf{D}_\ell)$ and $\mathrm{Orb}^{\mathrm{odd}}(\mathbf{D}_\ell)$ of the even and odd σ-orbits, respectively. We compute also the cardinalities

$$h^{\mathrm{even}}(\mathbf{D}_\ell) = \#\mathrm{Orb}^{\mathrm{even}}(\mathbf{D}_\ell) \quad \text{and} \quad h^{\mathrm{odd}}(\mathbf{D}_\ell) = \#\mathrm{Orb}^{\mathrm{odd}}(\mathbf{D}_\ell).$$

We compute $\mathrm{Orb}^{\mathrm{even}}(\mathbf{D}_\ell)$. Recall that $\ell = 2k$. For representatives of even σ-orbits we take

$$\begin{smallmatrix}1&&&&1\\&0&\cdots&0&\\0&&&&0\end{smallmatrix} \qquad \begin{smallmatrix}0&&&&0\\&0&\cdots&0&\\1&&&&1\end{smallmatrix} \qquad \begin{smallmatrix}2&&&&0\\&0&\cdots&0&\\0&&&&0\end{smallmatrix} \qquad \begin{smallmatrix}0&&&&0\\&0&\cdots&0&\\2&&&&0\end{smallmatrix}$$

and for each integer j with $0 < 2j \le k$, the 2-labeling with 1 at $2j$. Thus

$$h^{\text{even}}(\mathbf{D}_{2k}) = \lfloor k/2 \rfloor + 4.$$

We compute $\mathrm{Orb}^{\text{odd}}(\mathbf{D}_\ell)$. For representatives of odd σ-orbits we take

$$\begin{smallmatrix} 1 & & 0 \\ & 0\cdots 0 & \\ 1 & & 0 \end{smallmatrix} \qquad \begin{smallmatrix} 1 & & 0 \\ & 0\cdots 0 & \\ 0 & & 1 \end{smallmatrix}$$

and for each integer j with $1 < 2j+1 \le k$, the 2-labeling with 1 at $2j+1$. Thus

$$h^{\text{odd}}(\mathbf{D}_{2k}) = \lceil k/2 \rceil + 1.$$

As an example, we give a list of representatives of even and odd orbits for $\mathbf{D}_6$:

$$\mathrm{Orb}^{\text{even}}(\mathbf{D}_6): \quad \begin{smallmatrix} 1 & & 1 \\ & 0\,0\,0 & \\ 0 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} 0 & & 0 \\ & 0\,0\,0 & \\ 1 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} 2 & & 0 \\ & 0\,0\,0 & \\ 0 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} 0 & & 0 \\ & 0\,0\,0 & \\ 2 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} 0 & & 0 \\ & 1\,0\,0 & \\ 0 & & 0 \end{smallmatrix},$$

$$\mathrm{Orb}^{\text{odd}}(\mathbf{D}_6): \quad \begin{smallmatrix} 1 & & 0 \\ & 0\,0\,0 & \\ 1 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} 1 & & 0 \\ & 0\,0\,0 & \\ 0 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} 0 & & 0 \\ & 0\,1\,0 & \\ 0 & & 0 \end{smallmatrix}.$$

Note that if $\ell > 4$, our compact half-spin group $\mathbf{G}$ has no outer automorphisms, hence all its real forms are *inner* forms, and we have computed the Galois cohomology for all the forms of $\mathbf{G}$.

Note also that for the compact half-spin group $\mathbf{G}$ we have

$$\#H^1(\mathbb{R}, \mathbf{G}) = h^{\text{even}}(\mathbf{D}_{2k}) = \lfloor k/2 \rfloor + 4 = \lfloor \ell/4 \rfloor + 4.$$

For comparison, $\#H^1(\mathbb{R}, \mathbf{SO}_{2\ell}) = \ell + 1$. We have $\lfloor \ell/4 \rfloor + 4 = \ell + 1$ for an even natural number ℓ if and only if $\ell = 4$. (In this case, because of triality, our half-spin group $\mathbf{G}$ is isomorphic to $\mathbf{SO}_8$.)

References

[A] J. Adams, *Galois and θ cohomology of real groups,* `arXiv:1310.7917[math.GR]`.

[B1] M. V. Borovoi, *Galois cohomologies of real reductive groups, and real forms of simple Lie algebras*, Functional Anal. Appl. **22**:2 (1988), 135–136.

[B2] M. Borovoi, *Galois cohomology of reductive algebraic groups over the field of real numbers,* `arXiv:1401.5913[math.GR]`.

[BE] M. Borovoi and Z. Evenor, *Galois cohomology of simply connected real groups, and labelings of Dynkin diagrams*, `arXiv:1406.4362[math.GR]`.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie, Chap. IV–VI,* Hermann, Paris, 1968.

[C] B. Conrad, *Non-split reductive groups over* **Z**, to appear in: "Autour des schémas en groupes (Group Schemes, A celebration of SGA3)", vol. 2, Soc. Math. France, `http://math.stanford.edu/~conrad/papers/redgpZsmf.pdf`.

[K] V. G. Kac, *Automorphisms of finite order of semisimple Lie algebras,* Functional Anal. Appl., **3**:3 (1969), 252–254.

[OV1] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups,* Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990.

[OV2] A. L. Onishchik and E. B. Vinberg (Eds.), *Lie Groups and Lie Algebras III*, Encyclopaedia of Mathematical Sciences, Vol. 41, Springer-Verlag, Berlin, 1994.

[S] J.-P. Serre, *Cohomologie galoisienne*, LNM 5, 5-ème ed., Springer-Verlag, Berlin, 1994.

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